Well-posed forms of the 3+1 conformally-decomposed Einstein equations *

Simonetta Frittelli[a †], Oscar A. Reula[b ‡]

[a] *Physics Department, Duquesne University, Pittsburgh, PA 15282 USA*

[b] *FaMAF, Universidad Nacional de Córdoba, Ciudad Universitaria, 5000 Córdoba, Argentina*

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We show that well-posed, conformally-decomposed formulations of the 3+1 Einstein equations can be obtained by densitizing the lapse and by combining the constraints with the evolution equations. We compute the characteristics structure and verify the constraint propagation of these new well-posed formulations. In these formulations, the trace of the extrinsic curvature and the determinant of the 3-metric are singled out from the rest of the dynamical variables, but are evolved as part of the well-posed evolution system. The only free functions are the lapse density and the shift vector. We find that there is a 3-parameter freedom in formulating these equations in a well-posed manner, and that part of the parameter space found consists of formulations with causal characteristics, namely, characteristics that lie only within the lightcone. In particular there is a 1-parameter family of systems whose characteristics are either normal to the slicing or lie along the lightcone of the evolving metric.

PACS

I. INTRODUCTION

Analytical work in recent years has produced a number of systems of evolution equations which are equivalent to the Einstein equations at the constraint manifold, and which have a well posed initial value formulation [1–7].

What motivates interest in this type of result is a general understanding (see for instance [8]) that explicit well-posedness would be relevant in implementing consistent and stable numerical algorithms to integrate blackhole space-times.

The well-posed schemes for which a numerical code has been implemented appear not to exibit significant improvements over other methods, there being several factors relevant to numerical implementation which play a significant role. What is puzzling, however, is that, on the other hand, there have been numerical simulations with apparently better behavior, but which are based on systems which do not seem to have the well-posed character. One preponderant feature of these numerically more robust schemes is that they are built on a decomposition of the intrinsic metric into a metric of unit determinant and the determinant itself, and of the extrinsic curvature into trace and trace-free part. With slight variations, this way of evolving the 3+1 Einstein equations has been considered by [9,10]. Quite recently, this form has been shown to possess striking computational advantages over the standard form [11]. We refer to this general scheme as a conformally-decomposed formulation of the 3+1 Einstein equations.

It is difficult to explain the success of these systems as opposed to the well-posed evolution schemes, or even to the standard (ADM) evolution schemes. The relative sizes of the fields of well-posed systems are roughly the same for different spectral frequencies in a Fourier representation, which helps explain the stability of the system via numerical analysis. However, in the conformally-decomposed systems this does not happen in general (for standard norms), as it does not happen for the standard ADM system, thus making it more difficult to justify their relative better behavior.

We can speculate on two features that can possibly bear relevance to well behaved numerical evolution. One feature is that good behavior in evolution is related to constraint violations. If the system preserves more accurately the constraints, then the evolution remains closer to the constraint submanifold, which contains the physical solutions. Outside this submanifold the solutions are unphysical; thus, there is no compelling reason to rule out fast growths for seemingly tame initial data for unphysical solutions. Thus, we suggest that controlling the constraint violations may lead to well-behaved numerical evolution. In this respect, it has been shown [12], that (at least in the linearized case) there are well-posed modifications of the Einstein equations outside the constraint submanifold which make that

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†e-mail: simo@orion.physics.duq.edu

‡e-mail: reula@fis.uncor.edu

submanifold an attractor, thus improving the chances of building numerical codes with better behaved constraint propagation. Another possible cause of concern for generating numerical instabilities is the nature of the boundary conditions which are ussually imposed. The initial-boundary value problem for the Einstein equations has not been studied for the systems used in numerical simulations and where instabilities have been found (see nevertheless [13] for a complete theory of boundary values for Einstein's equations in conformal frame variables, and [14] for a linearized study), thus the set of boundary conditions for which the constraint equations are satisfied is in general not known. In dealing with this problem, establishing well-posedness for the Cauchy problem is a necessary first step.

The other feature which could give rise to numerical instabilities is the relative sizes of the "longitudinal" and the "radiative" modes in general relativity. In all non-trivial asymptotically flat solutions (either vacuum or with matter satisfying the appropiate energy conditions) the positivity of the mass implies the existence of longitudinal modes, and there are many astrophysically relevant cases where there is an approximate local notion of longitudinal versus transverse modes, and where the former are several orders of magnitude bigger that the latter. If they are not properly separated in the numerical algorithms, the errors caused by finite differencing might be of the order of the "radiative" modes, and bad behavior can be expected. The separation of the conformal freedom in the more successful codes can perhaps be thought of as a way of dealing with this issue, or at least isolating it.

In this work, we focus on this latter aspect. A technique for taking advantage of the conformal factor to partially decouple the "longitudinal" and "transversal" modes was used to obtain results on the Newtonian limit of general relativity [2]. In that case the conformal field was fixed via an elliptic equation, decoupling in this way the more prominent Newtonian potential to first order from the radiative degrees of freedom. Further studies on this problem would be critical to obtain realistic simulations of most astrophysically relevant problems.

Here we construct 3-parameter families of first-order well-posed systems wich share some of the properties of the more successful systems, such as the conformal decomposition of the fundamental fields, in the hope that their study would help understand what is causing them to behave better than others. In Section II we apply techniques similar to those we used in [5,7] in order to obtain versions of the 3+1 equations that are conformally-decomposed but which are well posed. Additionally, we calculate the structure of characteristics and show that for a open region in parameter space the resulting equations are metric-causal, namely they have all propagation cones inside or coincident with the light cone. There is even a one parameter subfamily for which propagation is either along the light cone or normal to the slices.

Furthermore we show that the constraints are propagated by these well-posed evolution equations. As oposed to [15], where also analytical studies of systems with this decomposition have been done, in this work the trace of the extrisic curvature and the determinant of the intrinsic metric are considered dynamical variables and are evolved jointly with the rest of the system.

II. SYSTEM II

The conformally-decomposed system that we take as starting point has appeared in [11], and is a variation of the system used by Shibata and Nakamura [10]. It is a system of 15 equations for 15 variables $(\phi, K, \tilde{\gamma}_{ij}, \tilde{A}_{ij}, \tilde{\Gamma}^i)$, and is referred to as System II in [11], to distinguish it from the standard 3+1 Einstein equations [16]. These variables are related to the intrinsic metric γ_{ij} and extrinsic curvature K_{ij} as follows:

$$e^{4\phi} = \det(\gamma_{ij})^{1/3} \tag{1a}$$

$$\tilde{\gamma}_{ij} = e^{-4\phi}\gamma_{ij} \tag{1b}$$

$$K = \gamma^{ij}K_{ij} \tag{1c}$$

$$\tilde{A}_{ij} = e^{-4\phi}\left(K_{ij} - \frac{1}{3}\gamma_{ij}K\right) \tag{1d}$$

$$\tilde{\Gamma}^i = -\tilde{\gamma}^{ij}{}_{,j} \tag{1e}$$

where $\tilde{\gamma}^{ij}$ is the inverse of $\tilde{\gamma}_{ij}$. The Einstein equations in terms of these variables are equivalent to the following:

$$\frac{d}{dt}\phi = -\frac{1}{6}\alpha K \tag{2a}$$

$$\frac{d}{dt}\tilde{\gamma}_{ij} = -2\alpha\tilde{A}_{ij} \tag{2b}$$

$$\frac{d}{dt}K = -\gamma^{ij}D_iD_j\alpha + \alpha\left(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2\right) + \frac{1}{2}\alpha(\rho + S) \tag{2c}$$

$$\frac{d}{dt}\tilde{A}_{ij} = e^{-4\phi}\left(-(D_iD_j\alpha)^{TF} + \alpha(R_{ij}^{TF} - S_{ij}^{TF})\right) + \alpha(K\tilde{A}_{ij} - 2\tilde{A}_{il}\tilde{A}^l{}_j) \tag{2d}$$

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{\Gamma}^i &= -2\tilde{A}^{ij}\alpha_j + 2\alpha\left(\tilde{\Gamma}^i_{jk}\tilde{A}^{kj} - \frac{2}{3}\tilde{\gamma}^{ij}K_{,j} - \tilde{\gamma}^{ij}S_j + 6\tilde{A}^{ij}\phi_{,j}\right) \\ &\quad -\frac{\partial}{\partial x^j}\left(\beta^l\tilde{\gamma}^{ij}{}_{,l} - 2\tilde{\gamma}^{m(j}\beta^{i)}{}_{,m} + \frac{2}{3}\tilde{\gamma}^{ij}\beta^l{}_{,l}\right).\end{aligned} \tag{2e}$$

Here α is the lapse function, β^i is the shift vector, and $\tilde{\Gamma}^i_{jk}$ are the connection coefficients of $\tilde{\gamma}_{ij}$. The superscript ^{TF} denotes trace-free part, e.g. $R_{ij}^{TF} = R_{ij} - \gamma_{ij}R/3$. Indices are raised and lowered with $\tilde{\gamma}^{ij}$ and its inverse. We use the shorthand notation

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} - \pounds_\beta \tag{2f}$$

where $\pounds_\beta$ is the Lie derivative along β^i. We have, as well,

$$R_{ij}^{TF} = R_{ij} - \frac{1}{3}\gamma^{ij}\gamma^{kl}R_{kl} \tag{2g}$$

$$R_{ij} = \tilde{R}_{ij} + R_{ij}^{\phi} \tag{2h}$$

$$R_{ij}^{\phi} = -2\tilde{D}_i\tilde{D}_j\phi - 2\tilde{\gamma}_{ij}\tilde{\gamma}^{kl}\tilde{D}_k\tilde{D}_l\phi + 4\tilde{D}_i\phi\tilde{D}_j\phi - 4\tilde{\gamma}_{ij}\tilde{\gamma}^{kl}\tilde{D}_k\phi\tilde{D}_l\phi \tag{2i}$$

$$\begin{aligned}\tilde{R}_{ij} &= -\frac{1}{2}\tilde{\gamma}^{kl}\tilde{\gamma}_{ij,kl} + \tilde{\gamma}_{k(i}\tilde{\Gamma}^k{}_{,j)} \\ &\quad + \tilde{\Gamma}^k\tilde{\Gamma}_{(ij)k} + \tilde{\gamma}^{lm}\left(2\tilde{\Gamma}^k{}_{l(i}\tilde{\Gamma}_{j)km} + \tilde{\Gamma}^k{}_{il}\tilde{\Gamma}_{kmj}\right)\end{aligned} \tag{2j}$$

$$\tilde{\Gamma}^k{}_{ij} = \frac{1}{2}\tilde{\gamma}^{kl}\left(\tilde{\gamma}_{il,j} + \tilde{\gamma}_{jl,i} - \tilde{\gamma}_{ij,l}\right). \tag{2k}$$

This system is first-order in time and second-order in space, thus it is of second order overall. We show how System II can be handled in order to be turned into a well-posed form. Firstly we reduce the system to a straightforward first order form, and subsequently we densitize the lapse and combine the constraints into the evolution equations.

A. System II reduced to first-order form

We define a set of 12 additional variables

$$V_{ijk} \equiv \tilde{\gamma}_{ij,k} - \frac{3}{5}\tilde{\gamma}_{k(i}\tilde{\gamma}_{j)n,s}\tilde{\gamma}^{ns} + \frac{1}{5}\tilde{\gamma}_{ij}\tilde{\gamma}_{kn,s}\tilde{\gamma}^{ns}, \tag{3a}$$

which is trace free in all its indices, namely: $V_{ijk}\tilde{\gamma}^{ij} = 0$ and $V_{ijk}\tilde{\gamma}^{jk} = V_{ijk}\tilde{\gamma}^{ik} = 0$, and another set of 3 additional variables

$$Q_i \equiv \phi_{,i}\,. \tag{3b}$$

Evolution equations for these new variables are obtained by taking a time derivative of (3) and commuting time and spatial derivatives in the resulting right-hand sides. The complete system of equations is now

$$\frac{d}{dt}\phi = -\frac{1}{6}\alpha K \tag{4a}$$

$$\frac{d}{dt}\tilde{\gamma}_{ij} = -2\alpha\tilde{A}_{ij} \tag{4b}$$

$$\frac{d}{dt}K = -\gamma^{ij}D_iD_j\alpha + \alpha\left(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2\right) + \frac{1}{2}\alpha(\rho + S) \tag{4c}$$

$$\frac{d}{dt}\tilde{A}_{ij} = e^{-4\phi}\left(-(D_iD_j\alpha)^{TF} + \alpha(R_{ij}^{TF} - S_{ij}^{TF})\right) + \alpha(K\tilde{A}_{ij} - 2\tilde{A}_{il}\tilde{A}^l{}_j) \tag{4d}$$

$$\dot{\tilde{\Gamma}}^i - \beta^l\tilde{\Gamma}^i{}_{,l} = -2\tilde{A}^{ij}\alpha_{,j} + 2\alpha\left(\tilde{\Gamma}^i{}_{jk}\tilde{A}^{kj} - \frac{2}{3}\tilde{\gamma}^{ij}K_{,j} - \tilde{\gamma}^{ij}S_j + 6\tilde{A}^{ij}Q_j\right) - \beta^l{}_{,j}\tilde{\gamma}^{ij}{}_{,l}$$

$$
+\tilde\Gamma^m\beta^i{}_{,m}+\tilde\gamma^{mi}{}_{,j}\,\beta^j{}_{,m}+2\tilde\gamma^{m(i}\beta^{j)}{}_{,mj}+\frac{2}{3}\tilde\Gamma^i\beta^l{}_{,l}+\frac{2}{3}\tilde\gamma^{ij}\beta^l{}_{,lj} \tag{4e}
$$

$$
\begin{aligned}
\dot V_{ijk}-\beta^l V_{ijk,l} &= -2\alpha\tilde A_{ij,k}+\frac{6}{5}\alpha\tilde\gamma_{k(i}\tilde A_{j)m,n}\tilde\gamma^{mn}-\frac{2}{5}\alpha\tilde\gamma_{ij}\tilde A_{km,n}\tilde\gamma^{mn}\\
&\quad+\beta^l{}_{,k}\,V_{ijl}+\beta^l{}_{,i}\,V_{ljk}+\beta^l{}_{,j}\,V_{ilk}-2\alpha_{,k}\,\tilde A_{ij} && \text{(4f)}\\
&\quad+\frac{6}{5}\tilde\gamma_{k(i}\tilde A^n_{j)}\alpha_{,n}+\frac{6}{5}\alpha\tilde A_{k(i}\tilde\Gamma_{j)}-\frac{2}{5}\alpha\tilde A_{ij}\tilde\Gamma_k && \text{(4g)}\\
&\quad+\tilde\gamma_{l(i}\beta^l{}_{,j)k}-\frac{3}{5}\beta^l{}_{,l(j}\,\tilde\gamma_{i)k}+\frac{1}{5}\tilde\gamma_{ij}\beta^l{}_{,lk} && \text{(4h)}\\
&\quad-\beta^l{}_{,ns}\,\tilde\gamma^{ns}\left(\frac{3}{5}\tilde\gamma_{k(i}\tilde\gamma_{j)l}-\frac{1}{5}\tilde\gamma_{ij}\tilde\gamma_{kl}\right) && \text{(4i)}\\
&\quad+\left(2\alpha\tilde A^{ns}+\tilde\gamma^{ns}\beta^s{}_{,m}\right)\left(\frac{3}{5}\tilde\gamma_{k(i}\tilde\gamma_{j)n,s}-\frac{1}{5}\tilde\gamma_{ij}\tilde\gamma_{ks,n}\right) && \text{(4j)}\\
\dot Q_i-\beta^l Q_{i,l} &= -\frac{1}{6}\alpha K_{,i}+\beta^l{}_{,i}\,Q_l-\frac{1}{6}\alpha_{,i}\,K && \text{(4k)}
\end{aligned}
$$

where, as before, $R^{TF}_{ij}=R_{ij}-\frac{1}{3}\gamma^{ij}\gamma^{kl}R_{kl}$, with $R_{ij}=\tilde R_{ij}+R^\phi_{ij}$, and

$$
\begin{aligned}
R^\phi_{ij} &= -2\tilde D_i Q_j-2\tilde\gamma_{ij}\tilde\gamma^{kl}\tilde D_k Q_l+4Q_iQ_j-4\tilde\gamma_{ij}\tilde\gamma^{kl}Q_kQ_l, && \text{(5a)}\\
\tilde R_{ij} &= -\frac{1}{2}V_{ijk,}{}^{k}+\frac{7}{10}\tilde\gamma_{k(i}\tilde\Gamma^k{}_{,j)}+\frac{1}{10}\tilde\gamma_{ij}\tilde\Gamma^k{}_{,k}\\
&\quad-\frac{3}{10}\left(\tilde\Gamma^k V_{k(ij)}+\tilde\Gamma_{(i}V_{j)k}{}^{k}+\frac{9}{10}\tilde\Gamma_i\tilde\Gamma_j\right)\\
&\quad-\frac{1}{5}\left(V_{ijk}\tilde\Gamma^k+\tilde\gamma_{ij}\tilde\Gamma^k V_{km}{}^{m}-\frac{1}{10}\tilde\gamma_{ij}\tilde\Gamma^k\tilde\Gamma_k\right)\\
&\quad+\tilde\Gamma^k\tilde\Gamma_{(ij)k}+2\tilde\Gamma^{kl}{}_{(i}\tilde\Gamma_{j)kl}+\tilde\Gamma^k{}_{il}\tilde\Gamma_{kj}{}^{l}, && \text{(5b)}\\
\tilde\Gamma^k{}_{ij} &= V^k{}_{(ij)}-\frac{1}{2}V_{ij}{}^{k}-\frac{1}{5}\delta^k_{(i}\tilde\Gamma_{j)}+\frac{2}{5}\tilde\gamma_{ij}\tilde\Gamma^k, && \text{(5c)}
\end{aligned}
$$

and indices are raised and lowered with $\tilde\gamma^{ij}$ and $\tilde\gamma_{ij}$ respectively. The derivatives of the form $\tilde\gamma_{ij,k}$ that appear in the right-hand sides of (4) must be interpreted simply as shorthands for combinations of the fields $\tilde\Gamma^i$ and V_{ijk}, via

$$
\tilde\gamma_{ij,k}=V_{ijk}+\frac{3}{5}\tilde\gamma_{k(i}\tilde\Gamma_{j)}-\frac{1}{5}\tilde\gamma_{ij}\tilde\Gamma_k \tag{6}
$$

For this first-order system to be equivalent to the Einstein equations (in the sense that its set of solutions is the same as that of the Einstein equations), the following set of constraints must be imposed on the initial data (and are subsequently preserved by the evolution, as will be shown in the next section):

$$
\begin{aligned}
\mathcal H &= \gamma^{ij}R_{ij}-\tilde A_{ij}\tilde A^{ij}+\frac{2}{3}K^2-2\rho && \text{(7a)}\\
\mathcal P_i &= \tilde\gamma^{jl}D_l\tilde A_{ij}-\frac{2}{3}D_iK+4Q_l\tilde A^l{}_i+\frac{4}{3}KQ_i-S_i && \text{(7b)}\\
\mathcal G^i &= \tilde\Gamma^i+\tilde\gamma^{ij}{}_{,j} && \text{(7c)}\\
\mathcal Q_i &= Q_i-\phi_{,i} && \text{(7d)}\\
\mathcal V_{ijk} &= V_{ijk}-\tilde\gamma_{ij,k}+\frac{3}{5}\tilde\gamma_{k(i}\tilde\gamma_{j)n,s}\tilde\gamma^{ns}-\frac{1}{5}\tilde\gamma_{ij}\tilde\gamma_{kn,s}\tilde\gamma^{ns}, && \text{(7e)}
\end{aligned}
$$

where

$$
\begin{aligned}
\gamma^{ij}R_{ij} &= e^{-4\phi}\Bigg(\tilde\Gamma^l{}_{,l}-8\tilde D^l Q_l-Q^lQ_l-\frac{1}{2}V_{ijl}V^{ijl}-\frac{15}{10}\tilde\Gamma^k V_{km}{}^{m}-\frac{1}{5}V^m{}_{m}k\tilde\Gamma^k && \text{(7f)}\\
&\quad-\frac{21}{100}\tilde\Gamma^k\tilde\Gamma_k+\tilde\Gamma^k\tilde\Gamma^m{}_{mk}+2\tilde\Gamma^{klm}\tilde\Gamma_{mkl}+\tilde\Gamma^{mkl}\tilde\Gamma_{mkl}\Bigg). && \text{(7g)}
\end{aligned}
$$

Constraints (7a) and (7b) are the hamiltonian and momentum constraints of the 3+1 decomposition of the Einstein equations, written in our choice of variables. Constraints (7c), (7d) and (7e) arise in turning the original second-order system into first order.

In (4) and (7), the derivative D_l is the covariant derivative with respect to γ_{ij}, and is related to $\tilde{D}_l$ by undifferentiated terms:

$$\Gamma^k{}_{ij} = \tilde{\Gamma}^k{}_{ij} + 2\left(Q_i\delta^k_j + Q_j\delta^k_i - Q_l\tilde{\gamma}^{kl}\tilde{\gamma}_{ij}\right) \tag{8}$$

B. Taking advantage of the available freedom

In this section we take advantage of two facts that have been used successfully in similar problems [17,5,7]. Firstly, we densitize the lapse α (and in doing so we introduce a free function, referred as "slicing density" in [18]):

$$\alpha = e^{4a\phi}\sigma \tag{9}$$

Like the shift vector β^i, the lapse density σ will be considered arbitrary but fixed, a source function independent of the dynamical fields.

Secondly, the evolution equations can be combined with the constraints without altering the set of solutions. We add the scalar constraint with a factor b to the evolution equation for K and we add the vector constraint to the evolution equations for $\tilde{\Gamma}^i$ and Q_i, with factors of c and d respectively. In this manner we obtain a system of the form

$$\dot{u} = \boldsymbol{A}^i(u)\nabla_i u + B(u). \tag{10}$$

A system of this form is known to be well posed if the matrix-valued vector $\boldsymbol{A}^i(u)$ admits a symmetrizer, namely, a positive definite, symmetric, bi-linear form $\boldsymbol{H}$, in the space of the fields u, whose product with $\boldsymbol{A}^i(u)$ yields a symmetric-bilinear-form-valued vector. Thus, in order to determine well-posedness, it suffices to consider the principal part of the system.

In this case, the principal terms are

$$\begin{aligned}
\dot{\phi} &= \beta^l\phi_{,l} && (11a)\\
\dot{\tilde{\gamma}}_{ij} &= \beta^l\tilde{\gamma}_{ij,l} && (11b)\\
\dot{K} &= \beta^l K_{,l} - \alpha(4a+8b)e^{-4\phi}\tilde{\gamma}^{kl}Q_{k,l} + \alpha b e^{-4\phi}\tilde{\Gamma}^l{}_{,l} && (11c)\\
\dot{\tilde{A}}_{ij} &= \beta^l\tilde{A}_{ij,l} + e^{-4\phi}\alpha\left(-\frac{1}{2}\tilde{\gamma}^{kl}V_{ijk,l} + \frac{7}{10}\left(\tilde{\gamma}_{k(i}\tilde{\Gamma}^k{}_{,j)} - \frac{1}{3}\tilde{\gamma}_{ij}\tilde{\Gamma}^k{}_{,k}\right)\right)\\
&\quad -2(2a+1)e^{-4\phi}\alpha\left(Q_{(i,j)} - \frac{1}{3}\tilde{\gamma}_{ij}\tilde{\gamma}^{kl}Q_{k,l}\right) && (11d)\\
\dot{\tilde{\Gamma}}^i &= \beta^l\tilde{\Gamma}^i{}_{,l} + \alpha c\tilde{A}^{il}{}_{,l} - \frac{2}{3}(c+2)\alpha\tilde{\gamma}^{il}K_{,l} && (11e)\\
\dot{V}_{ijk} &= \beta^l V_{ijk,l} - 2\alpha\tilde{A}_{ij,k} + \frac{6}{5}\alpha\tilde{\gamma}_{k(i}\tilde{A}_{j)m,n}\tilde{\gamma}^{mn} - \frac{2}{5}\alpha\tilde{\gamma}_{ij}\tilde{A}_{km,n}\tilde{\gamma}^{mn} && (11f)\\
\dot{Q}_i &= \beta^l Q_{i,l} - \frac{\alpha}{6}(1+4d)K_{,i} + d\alpha\tilde{\gamma}^{jl}\tilde{A}_{ij,l}. && (11g)
\end{aligned}$$

Our aim is to show that there exist choices of the numerical factors a, b, c, d such that the system (11) is symmetrizable, therefore, well posed. We establish this result by defining a candidate symmetrizer $\boldsymbol{H}$ given as

$$\begin{aligned}
\bar{u}\boldsymbol{H}u &= \phi^2 + \delta^{ik}\delta^{jl}\tilde{\gamma}_{ij}\tilde{\gamma}_{kl} + n_1^2 e^{-4\phi}K^2 + \tilde{A}^{ij}\tilde{A}_{ij} + n_2^2 e^{-4\phi}\tilde{\Gamma}^i\tilde{\Gamma}_i + \frac{e^{-4\phi}}{4}V_{ijk}V^{ijk}\\
&\quad + n_3^2 e^{-4\phi}Q^iQ_i
\end{aligned} \tag{12}$$

where n_1, n_2, n_3 are any fixed real numbers different from zero and bounded, so that $C^{-1}\boldsymbol{I} \leq \boldsymbol{H} \leq C\boldsymbol{I}$ where C is a positive constant and $\boldsymbol{I}$ is the identity operator on the space of u .

We can easily arrange the values of a, b, c, d so that symmetry of $\boldsymbol{H}\boldsymbol{A}^i(u)$ is attained. To this effect, they must satisfy

$$\frac{7}{10} = n_2^2 c \tag{13a}$$
$$-2(2a+1) = n_3^2 d \tag{13b}$$
$$-(4a+8b)n_1^2 = -\frac{1}{6}(1+4d)n_3^2 \tag{13c}$$
$$n_1^2 b = -\frac{2}{3}n_2^2(c+2) \tag{13d}$$

There is clearly plenty of freedom in the choice of a, b, c, d, since any choice that results in non-vanishing n_1, n_2, n_3 is allowed. The freedom is thus parametrized by the values of n_1^2, n_2^2, n_3^2, since these can take independent positive values. Thus our four parameters a, b, c, d are not all independent, but there is a relationship between them that reduces the freedom to 3 independent parameters. We can solve (13) for a, b, c, d in terms of n_1, n_2, n_3, which yields:

$$a = \frac{9/5 + 8n_2^2 + n_3^2/8}{(3n_1^2+2)} \tag{14a}$$
$$b = -\frac{2}{3n_1^2}\left(\frac{7}{10} + 2n_2^2\right) \tag{14b}$$
$$c = \frac{7}{10n_2^2} \tag{14c}$$
$$d = -\frac{2}{n_3^2}\left(2\frac{9/5 + 8n_2^2 + n_3^2/8}{(3n_1^2+2)} + 1\right) \tag{14d}$$

It is clear from (14) that a and c will take only strictly positive values, and b and d will take only strictly negative values, for all real values of n_1, n_2, n_3 different from zero.

C. Structure of characteristics

The system (11) is of the form

$$\boldsymbol{A^a}\frac{\partial u}{\partial x^a} = 0. \tag{15}$$

The characteristic covectors are covectors $\xi_a = (\xi_i, -v)$ such that $\xi_i \xi_j \gamma^{ij} = 1$ and such that

$$\det(\boldsymbol{A^a}\xi_a) = 0 \tag{16}$$

The values of v that satisfy (16) for every direction ξ_i are the characteristic speeds in that direction. In order to find these values we set up an eigenvalue problem for the principal symbol $\boldsymbol{A^a}\xi_a$ and find the null eigenvectors. The eigenvalue problem is

$$n^a \xi_a \phi = 0 \tag{17a}$$
$$n^a \xi_a \tilde{\gamma}_{ij} = 0 \tag{17b}$$
$$n^a \xi_a K = -(4a+8b)e^{-4\phi}\xi^k Q_k + b e^{-4\phi}\xi_l \tilde{\Gamma}^l \tag{17c}$$
$$\begin{aligned} n^a \xi_a \tilde{A}_{ij} = {} & e^{-4\phi}\left(-\frac{1}{2}\xi^k V_{ijk} + \frac{7}{10}\left(\tilde{\gamma}_{k(i}\xi_{j)}\tilde{\Gamma}^k - \frac{1}{3}\tilde{\gamma}_{ij}\xi_k \tilde{\Gamma}^k\right)\right) \\ & -2(2a+1)e^{-4\phi}\left(\xi_{(j}Q_{i)} - \frac{1}{3}\tilde{\gamma}_{ij}\xi^k Q_k\right) \end{aligned} \tag{17d}$$
$$n^a \xi_a \tilde{\Gamma}^i = c\xi_l \tilde{A}^{il} - \frac{2}{3}(c+2)\xi^i K \tag{17e}$$
$$n^a \xi_a V_{ijk} = -2\xi_k \tilde{A}_{ij} + \frac{6}{5}\tilde{\gamma}_{k(i}\tilde{A}_{j)m}\xi^m - \frac{2}{5}\tilde{\gamma}_{ij}\xi^m \tilde{A}_{km} \tag{17f}$$
$$n^a \xi_a Q_i = -\frac{1}{6}(1+4d)\xi_i K + d\xi^j \tilde{A}_{ij} \tag{17g}$$

Clearly, $n^a \xi_a = 0$ allows for 18 eigenvectors. This is because (17e), (17f) and (17g) in this case constitute an overdetermined system of 18 homogeneous equations for 6 unknowns $(\tilde{A}_{ij}, K)$, with zero as the only solution, whereas

(17c) and (17d) constitute a system of 6 equations for 18 variables, which leaves out 12 of the 18 fields $(V_{ijk}, \tilde{\Gamma}^i, Q)$ free. Lastly, (17a) and (17b) leave the 6 variables $(\tilde{\gamma}_{ij}, \phi)$ free. If we represent the eigenvectors in the form

$$(\tilde{\gamma}_{ij}, \phi, Q_i, \tilde{\Gamma}^{(L)}, \tilde{\Gamma}^{(T)}_i, V^{(L)}_{ij}, V^{(T)}_{ijk}, K, \tilde{A}^{(LL)}, \tilde{A}^{(LT)}_i, \tilde{A}^{(TT)}_{ij}) \tag{18}$$

where $\tilde{\Gamma}^{(L)} := \tilde{\Gamma}^i\xi_i,\ \tilde{\Gamma}^{(T)}_i := \tilde{\Gamma}_i - e^{-4\phi}\xi_i\tilde{\Gamma}^k\xi_k,\ V^{(L)}_{ij} := V_{ijk}\xi^k, V^{(T)}_{ijk} := V_{ijk} - e^{-4\phi}\xi_k V_{ijl}\xi^l,\ \tilde{A}^{(LL)} := \tilde{A}^{ij}\xi_i\xi_j,\ \tilde{A}^{(LT)}_i := \tilde{A}_{ij}\xi^j - e^{-4\phi}\xi_i\tilde{A}^{kl}\xi_k\xi_l,\ \tilde{A}^{(TT)}_{ij} := \tilde{A}_{ij} - 2e^{-4\phi}\xi_{(i}\tilde{A}_{j)l}\xi^l + e^{-8\phi}\xi_i\xi_j\tilde{A}^{kl}\xi_k\xi_l,$ then we have 5 eigenvectors corresponding to the five components of the conformal metric

$$(\tilde{\gamma}_{ij}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0); \tag{19a}$$

we have the determinant as an eigenfield

$$(0, \phi, 0, 0, 0, 0, 0, 0, 0, 0, 0); \tag{19b}$$

we have 7 eigenvectors corresponding to the seven transverse components of V_{ijk}

$$(0, 0, 0, 0, 0, 0, V^{(T)}_{ijk}, 0, 0, 0); \tag{19c}$$

we have 3 eigenvectors corresponding essentially to the three components of Q_i

$$(0, 0, Q_i, \frac{4a+8b}{b}\xi^i Q_i, 0, -2(2a+1)\xi_{(i}Q_{j)} - \frac{1}{3}\tilde{\gamma}_{ij}\left(\frac{7(4a+8b)}{10b} - 2(2a+1)\right)Q_l\xi^l, 0, 0, 0, 0); \tag{19d}$$

and we have 2 eigenvectors corresponding essentially to the two components of the transverse part of $\tilde{\Gamma}_i$

$$(0, 0, 0, 0, \tilde{\Gamma}^{(T)}_i, \frac{7}{10}\xi_{(i}\tilde{\Gamma}^{(T)}_{j)}, 0, 0, 0, 0); \tag{19e}$$

If $n^a\xi_a \neq 0$, then $\tilde{\gamma}_{ij} = \phi = 0$, and we can solve (17e), (17f) and (17g) for $(V_{ijk}, \tilde{\Gamma}^i, Q)$ in terms of ξ_a and $(\tilde{A}_{ij}, K)$. We can substitute $(V_{ijk}, \tilde{\Gamma}^i, Q)$ into (17c) and (17d), obtaining thus a system of 6 equations for the 6 variables $(\tilde{A}_{ij}, K)$ as follows

$$0 = Ke^{4\phi}\left((n^a\xi_a)^2 - \frac{1}{6}(4a+8b)(1+4d) + \frac{2}{3}b(c+2)\right) + \xi\cdot\tilde{A}\cdot\xi\,((4a+8b)d - bc) \tag{20a}$$

$$\begin{aligned} 0 = {} & \tilde{A}_{ij}e^{4\phi}\left(1 - (n^a\xi_a)^2\right) - \frac{K}{3}\left(\frac{7}{5}(c+2) - (2a+1)(1+4d)\right)\left(\xi_i\xi_j - \frac{1}{3}e^{4\phi}\tilde{\gamma}_{ij}\right) \\ & + \left(\frac{7c}{10} - \frac{3}{5} - 2d(2a+1)\right)\left(\xi^l\tilde{A}_{l(i}\xi_{j)} - \frac{1}{3}\tilde{\gamma}_{ij}\xi\cdot\tilde{A}\cdot\xi\right) \end{aligned} \tag{20b}$$

where we have used the notation

$$\xi\cdot\tilde{A}\cdot\xi := \xi_i\tilde{A}^{ij}\xi_j. \tag{21}$$

If $1-(n^a\xi_a)^2 = 0$, then $K = \xi{\cdot}\tilde{A}\xi = 0$ by (20a), which implies $\xi^l\tilde{A}_{li} = 0$ by (20b). However, two of the five components of $\tilde{A}_{ij}$ are thus free, which means that there are 4 eigenvectors, essentially labeled by the transverse components of $\tilde{A}_{ij}$. We have 2 eigenvectors for $n^a\xi_a = 1$

$$(0, 0, 0, 0, 0, -2\tilde{A}^{(TT)}_{ij}, 0, 0, 0, 0, \tilde{A}^{(TT)}_{ij}); \tag{22a}$$

and 2 eigenvectors for $n^a\xi_a = -1$

$$(0, 0, 0, 0, 0, 2\tilde{A}^{(TT)}_{ij}, 0, 0, 0, 0, \tilde{A}^{(TT)}_{ij}). \tag{22b}$$

If $1 - (n^a\xi_a)^2 \neq 0$, then contracting (20b) with ξ^i yields

$$0 = e^{4\phi}\xi^l \tilde{A}_{lj}\left(1+\frac{1}{2}\left(\frac{7c}{10}-\frac{3}{5}-2d(2a+1)\right)-(n^a\xi_a)^2\right)$$
$$-\frac{2}{9}e^{4\phi}K\xi_j\left(\frac{7}{5}(c+2)-(2a+1)(1+4d)\right)$$
$$+\frac{1}{6}\xi\cdot\tilde{A}\cdot\xi\;\xi_j\left(\frac{7}{10}c-\frac{3}{5}-2d(2a+1)\right) \tag{23}$$

Thus, if $1+\frac{1}{2}(7c/10-3/5-2d(2a+1))-(n^a\xi_a)^2=0$, then $K=\xi\cdot\tilde{A}\xi=0$ by (20a), which implies that (23) is identically satisfied, thus three out of the five equations (20b) are identities, the remaining two determining two components of $\tilde{A}_{ij}$. Thus two of the five components of $\tilde{A}_{ij}$ are free, which means that there are 4 eigenvectors, essentially labeled by the two longitudinal-transverse components of $\tilde{A}_{ij}$. We have 2 eigenvectors for $n^a\xi_a=\sqrt{(3/5-7c/10-2d(2a+1))/2}$, namely

$$(0,0,\frac{d}{C_1}\tilde{A}^{(LT)}_i,0,\frac{c}{C_1}\tilde{A}^{(LT)}_i,-\frac{4}{5C_1}\xi_{(i}\tilde{A}^{(LT)}_{j)},\frac{6}{5C_1}\left(\tilde{\gamma}_{k(i}\tilde{A}^{(LT)}_{j)}-\xi_k\xi_{(i}\tilde{A}^{(LT)}_{j)}-\frac{1}{3}\tilde{\gamma}_{ij}\tilde{A}^{(LT)}_k\right),0,0,\tilde{A}^{(LT)}_i,0), \tag{24a}$$

and 2 eigenvectors for $n^a\xi_a=-\sqrt{(3/5-7c/10-2d(2a+1))/2}$, namely

$$(0,0,-\frac{d}{C_1}\tilde{A}^{(LT)}_i,0,-\frac{c}{C_1}\tilde{A}^{(LT)}_i,\frac{4}{5C_1}\xi_{(i}\tilde{A}^{(LT)}_{j)},-\frac{6}{5C_1}\left(\tilde{\gamma}_{k(i}\tilde{A}^{(LT)}_{j)}-\xi_k\xi_{(i}\tilde{A}^{(LT)}_{j)}-\frac{1}{3}\tilde{\gamma}_{ij}\tilde{A}^{(LT)}_k\right),0,0,\tilde{A}^{(LT)}_i,0), \tag{24b}$$

where we have used the shorthand notation

$$C_1 := \sqrt{(3/5-7c/10-2d(2a+1))/2} \tag{25}$$

But if $1+\frac{1}{2}(7c/10-3/5-2d(2a+1))-(n^a\xi_a)^2\neq 0$, then $\xi^l\tilde{A}_{lj}$ is determined by the values of K and $\xi\cdot\tilde{A}\cdot\xi$ by (23), and if plugged back into (20b) it follows that all the components of $\tilde{A}_{ij}$ are determined by K and $\xi\cdot\tilde{A}\cdot\xi$. Therefore it is necessary that K and $\xi\cdot\tilde{A}\cdot\xi$ be nonvanishing. Contracting (23) with ξ^j we obtain

$$0=-\frac{2}{9}e^{4\phi}K\left(\frac{7}{5}(c+2)-(2a+1)(1+4d)\right)$$
$$+\xi\cdot\tilde{A}\cdot\xi\left(1+\frac{2}{3}\left(\frac{7}{10}c-\frac{3}{5}-2(2a+1)d\right)-(n^a\xi_a)^2\right) \tag{26}$$

Equations (20a) and (26) form a system of two homogeneous equations for K and $\xi\cdot\tilde{A}\cdot\xi$. Thus, for K and $\xi\cdot\tilde{A}\cdot\xi$ to be nonvanishing, it is necessary that the determinant of the system be zero. The determinant is

$$\frac{1}{45}(-3(n^a\xi_a)^2+2a)(15(n^a\xi_a)^2-9+10bc-80bd+20d-7c) \tag{27}$$

It can be seen that, because a and c are strictly positive and b and d are strictly negative, the four roots of the determinant are real. For the roots $n^a\xi_a$ of the determinant, we have

$$K=\frac{9}{2}\frac{1-(n^a\xi_a)^2+\frac{2}{3}\left(\frac{7c}{10}-\frac{3}{5}-2d(2a+1)\right)}{\frac{7}{5}(c+2)-(2a+1)(1+4d))}e^{-4\phi}\xi\cdot\tilde{A}\cdot\xi \tag{28a}$$

$$Q_i=\left(d-\frac{3(1+4d)}{4}\frac{1-(n^a\xi_a)^2+\frac{2}{3}\left(\frac{7c}{10}-\frac{3}{5}-2d(2a+1)\right)}{\frac{7}{5}(c+2)-(2a+1)(1+4d))}\right)\frac{e^{-4\phi}}{n^a\xi_a}\xi_i\xi\cdot\tilde{A}\cdot\xi \tag{28b}$$

$$\tilde{\Gamma}^{(L)}=\left(c-3(c+2)\frac{1-(n^a\xi_a)^2+\frac{2}{3}\left(\frac{7c}{10}-\frac{3}{5}-2d(2a+1)\right)}{\frac{7}{5}(c+2)-(2a+1)(1+4d))}\right)\frac{\xi\cdot\tilde{A}\cdot\xi}{n^a\xi_a} \tag{28c}$$

$$\tilde{\Gamma}^{(T)}_i=0 \tag{28d}$$

$$V^{(L)}_{ij}=-\frac{9}{5n^a\xi_a}\left(e^{-4\phi}\xi_i\xi_j-\frac{1}{3}\tilde{\gamma}_{ij}\right)\xi\cdot\tilde{A}\cdot\xi \tag{28e}$$

$$V^{(T)}_{ijk}=-\frac{6}{5n^a\xi_a}\left(\xi_i\xi_j-\tilde{\gamma}_{k(i}\xi_{j)}\right)e^{-4\phi}\xi\cdot\tilde{A}\cdot\xi \tag{28f}$$

$$\tilde{A}^{(LT)}_i=0 \tag{28g}$$

$$\tilde{A}^{(TT)}_{ij}=\frac{e^{-4\phi}}{2}(e^{-4\phi}\xi_i\xi_j-\tilde{\gamma}_{ij})\xi\cdot\tilde{A}\cdot\xi \tag{28h}$$

These are clearly four distinct eigenvectors, since there are four distinct values of $n^a\xi_a$ given by the roots of the determinant (27). These can be thought as being labeled, essentially, by K or $\xi\cdot\tilde{A}\cdot\xi$ indistinctly, or we can associate one characteristic speed to K and the other one to $\xi\cdot\tilde{A}\cdot\xi$, as we prefer to do below.

Summarizing, we have characteristic speeds obtained from the following distinct values of $n^a\xi_a$:

a) $n^a\xi_a = 0$, timelike, with eigenfields (essentially) $\tilde{\gamma}_{ij}, \phi, Q_i, \tilde{\Gamma}^{(T)}, V^{(T)}_{ijk}$.

b) $n^a\xi_a = 1$, null, with eigenfields (essentially) $\tilde{A}^{(TT)}_{ij}$.

c) $n^a\xi_a = (1 + \frac{1}{2}(7c/10 - 3/5 - 2d(2a+1)))^{1/2} \equiv C_1$, with eigenfields (essentially) $\tilde{A}^{(TL)}_i$.

d) $n^a\xi_a = (2a/3)^{1/2} \equiv C_2$, with eigenfield (essentially) K

e) $n^a\xi_a = (3/5 - 2bc/3 + 16bd/3 - 4d/3 + 7c/15)^{1/2} \equiv C_3$, with eigenfield (essentially) $\tilde{A}^{(LL)}$

In the expressions for the characteristic speeds c),d) and e), the parameters a, b, c, d are given in terms of n_1, n_2, n_3 via (14). These speeds may be superluminal or causal, depending on the values of n_1, n_2, n_3. We can choose n_1, n_2, n_3 so that C_1, C_2 and C_3 are all equal to 1. This is achieved by setting

$$n_1^2 = \frac{4}{15}\frac{280n_2^2 + 49 + 400n_2^4}{60n_2^2 - 49} \tag{29a}$$

$$n_3^2 = \frac{6400n_2^2}{60n_2^2 - 49} \tag{29b}$$

for any value of n_2^2 greater than 49/60. This means that there is a one-parameter family of well-posed conformally-decomposed systems with "physical" characteristics. From the analytical point of view, there does not appear to exist an argument for singling out a preferred value of n_2. It is likely that a preferred value of n_2 will be dictated by optimal numerical behavior. The expressions for a, b, c, d in terms of n_2, with n_1 and n_3 as above (29), are as follows:

$$a = \frac{3}{2}, \tag{30a}$$

$$b = -\frac{60n_2^2 - 49}{4(7 + 20n_2^2)}, \tag{30b}$$

$$c = \frac{7}{10n_2^2}, \tag{30c}$$

$$d = -\frac{60n_2^2 - 49}{800n_2^2}. \tag{30d}$$

D. Propagation of the constraints

The propagation of the constraints can be calculated by taking a time derivative of each one of the constraint expressions, and subsequently using the evolution equations (11) to eliminate the time derivative of the fields in the right-hand side in favor of spatial derivatives, which recombine to yield back the constraints. We obtain

$$\dot{\mathcal{H}} = \beta^l\mathcal{H}_{,l} + (c - 8d)\alpha e^{-4\phi}\tilde{\gamma}^{kl}\mathcal{P}_{k,l} + \cdots \tag{31a}$$

$$\dot{\mathcal{P}}_i = \beta^l\mathcal{P}_{i,l} + \frac{\alpha}{6}(1 - 4b)\mathcal{H}_{,i} - \frac{\alpha}{2}e^{-4\phi}\left(\tilde{\gamma}^{jl}\left(\tilde{\gamma}^{kr}\mathcal{V}_{ijk,rl} - \frac{7}{10}\tilde{\gamma}_{im}\mathcal{G}^m{}_{,jl}\right) + \frac{1}{10}\mathcal{G}^m{}_{,mi}\right)$$
$$-2\alpha(2a+1)e^{-4\phi}\tilde{\gamma}^{jl}\mathcal{Q}_{[i,l]j} + \cdots \tag{31b}$$

$$\dot{\mathcal{G}}^i = \beta^l\mathcal{G}^i{}_{,l} + \cdots \tag{31c}$$

$$\dot{\mathcal{Q}}_i = \beta^l\mathcal{Q}_{i,l} + \cdots \tag{31d}$$

$$\dot{\mathcal{V}}_{ijk} = \beta^l\mathcal{V}_{ijk,l} + \cdots \tag{31e}$$

where $\cdots$ denotes undifferentiated terms proportional to the constraints themselves. To analyze the constraint propagation we proceed to turn (31) into first order by defining several sets of variables which represent all the spatial derivatives of $\mathcal{V}_{ijk}, \mathcal{G}^i$ and $\mathcal{Q}_i$:

$$
\begin{aligned}
\mathcal{W}_{ij} &= \mathcal{V}_{ijk},^{k} - \frac{1}{5}\mathcal{G}_{(i,j)} + \frac{1}{15}\tilde{\gamma}_{ij}\mathcal{G}^{k},_{k} && \text{(32a)}\\
\mathcal{X}_{ijkl} &= \mathcal{V}_{ijk,l} - \frac{1}{3}\tilde{\gamma}_{kl}\mathcal{V}_{ijm},^{m} && \text{(32b)}\\
\mathcal{U}_{ij} &= \mathcal{Q}_{[i,j]} && \text{(32c)}\\
\mathcal{Z}_{ij} &= \mathcal{Q}_{(i,j)} && \text{(32d)}\\
\mathcal{T}_{ij} &= \mathcal{G}_{[i,j]} && \text{(32e)}\\
\mathcal{J}_{ij} &= \mathcal{G}_{(i,j)} + \frac{30}{7}A\mathcal{Q}_{(i,j)} && \text{(32f)}
\end{aligned}
$$

where A is a constant which will be fixed shortly. Calculating the time derivative of these we obtain the resulting first-order system of evolution of the constraints:

$$
\begin{aligned}
\dot{\mathcal{H}} &= \beta^{l}\mathcal{H}_{,l} + \alpha(c-8d)e^{-4\phi}\mathcal{P}_{l},^{l} + \cdots && \text{(33a)}\\
\dot{\mathcal{P}}_{i} &= \beta^{l}\mathcal{P}_{i,l} + \frac{\alpha}{6}(1-4b)\mathcal{H}_{,i} - \frac{\alpha}{2}e^{-4\phi}\mathcal{W}_{il},^{l} - 2\alpha(2a+1)e^{-4\phi}\mathcal{U}_{il},^{l} \\
&\quad - A\alpha e^{-4\phi}\mathcal{Z}_{il},^{l} + \frac{7\alpha}{30}e^{-4\phi}\mathcal{J}_{il},^{l} + \frac{11\alpha}{30}e^{-4\phi}\mathcal{T}_{il},^{l} + \cdots && \text{(33b)}\\
\dot{\mathcal{W}}_{ij} &= \beta^{l}\mathcal{W}_{ij,l} - \frac{c\alpha}{5}\mathcal{P}_{(i,j)} + \frac{c\alpha}{15}\tilde{\gamma}_{ij}\mathcal{P}_{l},^{l} + \cdots && \text{(33c)}\\
\dot{\mathcal{X}}_{ijkl} &= \beta^{m}\mathcal{X}_{ijkl,m} + \cdots && \text{(33d)}\\
\dot{\mathcal{U}}_{ij} &= \beta^{l}\mathcal{U}_{ij} + \alpha d\mathcal{P}_{[i,j]} + \cdots && \text{(33e)}\\
\dot{\mathcal{Z}}_{ij} &= \beta^{l}\mathcal{Z}_{ij} + \alpha d\mathcal{P}_{(i,j)} + \cdots && \text{(33f)}\\
\dot{\mathcal{T}}_{ij} &= \beta^{l}\mathcal{T}_{ij} + \alpha c\mathcal{P}_{[i,j]} + \cdots && \text{(33g)}\\
\dot{\mathcal{J}}_{ij} &= \beta^{l}\mathcal{J}_{ij} + \alpha\left(c + \frac{30}{7}Ad\right)\mathcal{P}_{(i,j)} + \cdots && \text{(33h)}\\
\dot{\mathcal{G}}^{i} &= \beta^{l}\mathcal{G}^{i},_{l} + \cdots && \text{(33i)}\\
\dot{\mathcal{Q}}_{i} &= \beta^{l}\mathcal{Q}_{i,l} + \cdots && \text{(33j)}\\
\dot{\mathcal{V}}_{ijk} &= \beta^{l}\mathcal{V}_{ijk,l} + \cdots && \text{(33k)}
\end{aligned}
$$

For this system there is a symmetrizer given by

$$
\begin{aligned}
\bar{u}\boldsymbol{H}_{C}u = {}& e^{4\phi}\frac{(1-4b)}{6(c-8d)}\mathcal{H}^{2} + \mathcal{P}_{i}\mathcal{P}^{i} + e^{-4\phi}\frac{5}{2c}\mathcal{W}_{ij}\mathcal{W}^{ij} + \mathcal{X}_{ijkl}\mathcal{X}^{ijkl} - e^{-4\phi}\frac{2(2a+1)}{d}\mathcal{U}_{ij}\mathcal{U}^{ij} \\
& - e^{-4\phi}\frac{A}{d}\mathcal{Z}_{ij}\mathcal{Z}^{ij} + e^{-4\phi}\frac{11}{30c}\mathcal{T}_{ij}\mathcal{T}^{ij} + e^{-4\phi}\frac{7/30}{c+Ad30/7}\mathcal{J}_{ij}\mathcal{J}^{ij} \\
& + \mathcal{G}_{i}\mathcal{G}^{i} + \mathcal{Q}_{i}\mathcal{Q}^{i} + \mathcal{V}_{ijk}\mathcal{V}^{ijk}. && \text{(34)}
\end{aligned}
$$

Taking $A = \frac{7}{60}(-c/d)$, $\boldsymbol{H}_{C}$ is positive definite because, under the conditions (14), all the factors accompanying the squares of the fields are strictly positive. This shows that no additional restrictions on the ranges of the parameters a, b, c, d are necessary in order to have well posed constraint evolution.

III. CONCLUSION

We have derived a 3-parameter family of well posed versions of the conformally-decomposed 3+1 equations, perhaps amenable to successful numerical integration. One might object that there is no need for it in view of the results in [11], but we can argue rather strongly that these results may prove helpful in pinning-down the main cause of numerical instabilities. This well posed version requires the lapse to be proportional to the determinant of the intrinsic geometry of the surfaces, and requires combinations of the contraints with the evolution equations. The lapse density σ and the shift vector β^{i} are arbitrary non-dynamical variables, which means that they must be specified as free source functions. This well posed version uses the same variables as the original system (except for the addition of the first spatial derivatives of the densitized 3-metric, referred to as "conformal metric" by the authors in [11]). In addition, this well-posed version of the original equations propagates the constraints in a stable manner, which is relevant to

unconstrained evolution. We think that this is the least invasive way to turn the original conformally-decomposed system into a well posed one. In practice, a choice of the numerical parameters n_1, n_2, n_3 must be made. The characteristic speeds depend on this choice.

Optimal choices of the parameters n_1, n_2, n_3 for numerical evolution are those that ensure that the characteristics are all either null or timelike. With such a choice, the formulation would be suited to evolve blackhole spacetimes outside the event horizon. Among these choices, it has been suggested [19] that the preferred one would be the one for which the characteristics are all "physical", namely, either null or normal to the slices. We have shown that such a choice is possible for an arbitrary $n_2 > \sqrt{49/60}$.

The systems obtained in this work are not contained in our previous work [5,7]. The choice of variables in [5,7] is inadequate for decomposing the trace and trace free part of the extrinsic curvature, as well as for extracting the determinant of the 3-metric. This is clear from the fact that, in that work, the available parameters α and β are not allowed to take the value $-1/3$ without the argument breaking down.

The systems obtained here differ significantly from the system obtained in [15] by considering the trace of the extrisic curvature K and the determinant of the instrinsic metric (and its derivatives) as dynamical variables on equal footing with the rest, rather than as free source functions. Furthermore, we have obtained a 3-parameter family of systems, one system for each appropriate choice of n_1, n_2, n_3, whereas in [15] there is only one system which preserves the trace conditions. Additionally, we have separated the divergence of the intrinsic metric $\tilde{\Gamma}^i$ from the divergence-free part of the metric. This decomposition keeps up with the spirit of [11].

We have found that in obtaining these well-posed formulations the lapse must be proportional to some power of the determinant of the intrinsic metric, since the parameter a cannot take the value 0. This is similar to our findings in [2,5,7], as well as other notable cases [6,20,21,18]. In our present case this is remarkable, since we have used quite general energy norms.

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